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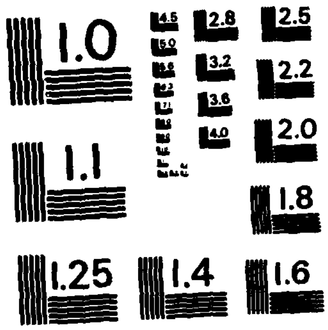
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MRC Technical Summary Report #2538

THE INTERFACES OF ONE-DIMENSIONAL FLOWS IN POROUS MEDIA

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July 1983

(Received June 8, 1983)

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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

The solutions of the equation $u_t = (u^m)_{xx}$ for $x \in \mathbb{R}$, $0 < t < T$, $m > 1$, where $u(x,0)$ is a nonnegative Borel measure that vanishes for $x > 0$ (and satisfies a growth condition at $-\infty$) exhibit a finite, monotone, continuous interface $x = \zeta(t)$ that bounds to the right the region where $u > 0$. We perform a detailed study of ζ: initial behaviour, waiting time, behaviour as $t \to \infty$. For certain initial data the solutions blow up in a finite time T^*: we calculate T^* in terms of $u(x,0)$ and describe the behaviour of ζ as $t \uparrow T^*$.

AMS (MOS) Subject Classifications: 35K65, 76S05, 35B40

Key Words: flows in porous media, interfaces, blow-up time, waiting time, asymptotic behaviour

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

~~The~~ porous media equation (PME)

$$\text{(PME)} \qquad u_t = (u^m)_{xx}, \quad (x,t) \in I \times (0,T)$$

where $m > 1$ and $T > 0$ are constants and I is an interval in $\mathbb{R}$ has been used as a model for a number of physical phenomena: heat diffusion at high temperatures, boundary layer theory, spread of a thin layer of viscous material and mainly the flow of gas in a porous medium. In all these applications $u \geq 0$.

The most distinctive characteristic of the solutions to (PME) as compared with the linear heat equation $u_t = u_{xx}$ is the finite speed of propagation: if the solution $u(x,t)$ is supported in a bounded interval $a < x < b$ at time $t = 0$ so it is for every time $t > 0$: $u(x,t) = 0$ for $x \notin (a',b')$. If we call $\zeta_1(t), \zeta_2(t)$ the best bounds a',b' at time t we obtain two monotone curves $x = \zeta_1(t)$, $x = \zeta_2(t)$ called interfaces that bound the support of the solution.

In this paper the properties of the interfaces are studied in terms of the initial data $u(x,0)$: it is assumed that $u(x,0) > 0$ and that $u(x,0) = 0$ for $x > 0$ being otherwise completely general and the study concentrates on $\zeta = \zeta_2(t)$. The behaviour of $\zeta(t)$ for very small times and very large times is shown to depend only on the behaviour of $u(x,0)$ near the interface and near $-\infty$ respectively and precise growth estimates are given. Also the occurrence of a blow up in finite time T^* is studied and estimated and the behaviour of $\zeta(t)$ as $t \to T^*$ described. Sometimes the interface is stationary for a certain time t^* and then begins to move: we characterize the existence of a positive waiting time t^* and give bounds for it.

Completing what was already known these results provide a satisfactory picture of the interfaces for the solutions of (PME) when $I = \mathbb{R}$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE INTERFACES OF ONE-DIMENSIONAL FLOWS IN POROUS MEDIA

Juan L. Vázquez[1]

INTRODUCTION

We consider the initial-value problem for the porous media equation

$$(P)\quad \begin{cases} u_t = (u^m)_{xx} & \text{in } Q_T = \mathbb{R} \times (0,T), \quad 0 < T \le \infty \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $m > 1$ is a physical constant and u_0 satisfies the assumptions:

(H1) $\quad u_0$ is a nonnegative Borel measure in $\mathbb{R}$, $u_0 \not\equiv 0$,

$$(H2)\qquad \sup_{R>1} R^{-\frac{m+1}{m-1}} \int_{|x|<R} du_0(x) < \infty ,$$

(H3) $\quad u_0$ vanishes on $(0,\infty)$.

The equation appears in a number of applications, the most typical being the flow of gas through a porous medium, where u stands for the density of the gas. This motivates the assumption (H1). Assumption (H2) is justified in view of the existence theory: in fact Bénilan, Crandall and Pierre [9] have constructed continuous weak solutions to the N-dimensional analogue: (P_N) $u_t = \Delta u^m$, $u(x,0) = u_0(x)$, $N \ge 1$, in a maximal strip $Q_{T^*} = \mathbb{R}^N \times (0,T^*)$, $0 < T^* = T^*(u_0) \le \infty$, under the condition

$$(H2')\qquad \sup_{R>1} R^{-(N+\frac{2}{m-1})} \int_{|x|<R} d|u_0|(x) < \infty$$

that reduces if $N = 1$, $u_0 \ge 0$, to (H2). Whenever $T^* < \infty$ we say that the solution

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blows up in finite time. They also prove that for nonnegative solutions a necessary and sufficient condition for global existence, i.e. $T^* = \infty$, is

$$\lim_{R\to\infty} R^{-(N+\frac{2}{m-1})} \int_{|x|<R} du_0(x) = 0 . \tag{0.1}$$

Also Aronson and Caffarelli [4] showed that every continuous, nonnegative weak solution of $u_t = \Delta u^m$ in a strip Ω_T, $T > 0$, has an initial trace $u(x,0)$ which is a locally bounded measure satisfying the growth condition (H2').

Recently Dahlberg and Kenig [11] proved that the continuous, nonnegative distributional solutions of (P_N) in Ω_T, $T > 0$, are unique. For another uniqueness result cf. [9] and its references. Early work on this subject goes back to Kalashnikov [14].

In view of these results (H2) is an optimal growth condition for the initial values of (P).

In this paper we are interested in describing the free-boundaries that appear in (P): indeed one of the most appealing features of (P) with $m > 1$ is the fact that when u_0 vanishes outside a compact interval then the support of $u(\cdot,t)$, $t > 0$, is also compact. This is called the finite propagation property and has been described by Oleinik, Kalashnikov and Czhou [17] in their 1958 paper where the existence, uniqueness, regularity and finite propagation for the solutions of (P) were first discussed at length.

The above considerations lead us to introduce the assumption (H3). We show that the solutions of (P) under the assumptions (H1)-(H3) vanish for large enough $x > 0$ for any fixed time $0 < t < T^*$. We define the outer right interface (or free boundary) of u as the curve $x = \zeta(t)$, where

$$\begin{cases} \zeta(t) = \sup\{x : u(x,t) > 0\} & \text{if } 0 < t < T^* \\ \zeta(0) = \sup\{x : \int_{(x,\infty)} du_0 > 0\} . \end{cases} \tag{0.2}$$

Then $\zeta : [0,T^*) \to [0,\infty)$ is a continuous, nondecreasing function and there is a time t^*,

$0 < t^* < T^*$, called <u>waiting-time</u>, such that $\zeta(t) = \zeta(0)$ if $0 < t < t^*$, $\zeta \in C^1(t^*,T^*)$ and $\zeta'(t) > 0$ if $t^* < t < T^*$. All these results were well-known when $u_0 \in L^1(\mathbb{R})$, cf. [1], [10], [16], [19]. In Section 1 we review these and other known results that we shall need and show that they continue to hold under the present conditions.

For simplicity we shall refer in the sequel to $x = \zeta(t)$ as <u>the</u> free boundary or interface. Remark that other interfaces may also appear: outer left interface and inner interfaces, cf. [19]. We shall make a brief comment on them at the end of the paper.

After Section 1 on preliminaries we estimate the blow-up time in terms of the growth of $|\int_0^x du_0(x)|$ as $x \to \infty$ in Section 2.

Section 3 is devoted to the waiting-time: we give necessary and sufficient conditions on u_0 for a positive waiting-time to exist as well as lower and upper estimates of it.

In Section 4 we construct a class of global self-similar solutions behaving as $x \to -\infty$ like $O(|x|^\alpha)$, $-1 < \alpha < 2/(m - 1)$.

In Section 5 we prove that the behaviour of the interface for small time depends only on the behaviour of u_0 near 0. By comparing with the explicit solutions of Section 4 we give rates of growth for small t for various classes of initial data.

A similar study is performed in Section 6 for large t. Now the behaviour of ζ depends on the behaviour of u_0 for large negative x. In particular if $u_0(x) \sim |x|^\alpha$ for an α : $-1 < \alpha < 2/(m - 1)$ as $x \to -\infty$ then as $t \to \infty$, $\zeta(t) \sim t^\gamma$ with $\gamma = (2 - \alpha(m - 1))^{-1}$.

Finally in Section 7 we study the behaviour of $\zeta(t)$ as $t \to T^*$ when T^* is finite. In particular we show conditions under which $\zeta(t) \uparrow \infty$ as $t \uparrow T^*$. We also study the blow-up set, i.e. the set of points $x \in \mathbb{R}$ for which $u(x,t) \to \infty$ as $t \to T^*$.

An interesting question not dealt with here is that of determining if $\zeta \in C^1(0,T^*)$. The only point where this may not be true is t^*. In [5] Aronson, Caffarelli and Kamin exhibited a class of initial data for which ζ is C^1 smooth. Recently Aronson, Caffarelli and the author [6] have proved that for roughly the complementary class of initial data $\zeta'(t+)$ is discontinuous at $t = t^*$. Self-similar solutions starting smoothly after a positive waiting time are constructed in [15].

1. PRELIMINARIES

1.1. Existence and uniqueness of solutions.

We recall here the results that we need from [9]:

THEOREM A. Let u_0 be a Borel measure satisfying (H2). Then there exists a maximal time $T^* \in (0,\infty]$ for which a solution u can be defined in $(0,T^*)$ such that

(i) $u \in C((0,T^*); L^1_{loc}(\mathbb{R})) \cap L^\infty_{loc}((0,T^*);X)$

(ii) $u(x,t)(1+|x|^2)^{-1/(m-1)} \in L^\infty_{loc}(\mathbb{R}\times(0,T^*))$

(iii) For $\psi \in C_0^\infty(\mathbb{R}\times[0,T^*))$ we have

$$\int_0^{T^*}\int (u\psi_t + u^m\psi_{xx})\,dxdt = \int \psi(x,0)\,du_0(x) .$$

Moreover if $T^* < \infty$ (in which case we say that the solution blows up in finite time)

(iv) $\lim_{t\uparrow T^*} |||u(\cdot,t)|||_1 = \infty$. $\square$

Here above X denotes the space of functions $f \in L^1_{loc}(\mathbb{R})$ such that

$$(1.1)\qquad |||f|||_r \sup_{R>r} R^{-\frac{m+1}{m-1}} \int_{|x|<R} |f|\,dx < \infty$$

for some (= all) $r > 0$, equipped with the norm $|||\cdot|||_1$. [9] contains further information on the solutions: uniqueness, ... In particular it is important to remark that the solution u with initial data $u_0 \geq 0$ can be obtained as the limit of solutions u_n with smooth, compactly supported initial data. Also P. Sacks proved that u is continuous in Q_{T^*}.

For uniqueness and comparison purposes we shall use the following result of Dahlberg and Kenig [11]:

THEOREM B. Let $u_1(x,t)$, $u_2(x,t)$ be continuous, nonnegative functions in a strip $Q_T = \mathbb{R}\times(0,T)$, $T>0$, such that

i) u_1 and u_2 are solutions of $u_t = (u^m)_{xx}$ in $\mathcal{D}'(Q_T)$;

ii) <u>the initial traces</u> $u_1(x,0)$, $u_2(x,0)$ (that exist thanks to [4]) <u>satisfy</u> $u_1(x,0) \le u_2(x,0)$ <u>as measures</u>.

<u>Then</u> $u_1(x,t) \le u_2(x,t)$ <u>in</u> Q_T. □

1.2. <u>Properties of the solutions</u>.

The following properties are valid for solutions with smooth initial data in $L^1(\mathbb{R})$ and remain valid for general initial data by approximation.

PROPERTY S1. (i) $(u^{m-1})_{xx} \ge -\frac{(m-1)}{m(m+1)t}$ <u>and</u> (ii) $u_t \ge -\frac{u}{(m+1)t}$ <u>in</u> $\mathcal{D}'(Q_T)$.

PROPERTY S2. <u>If</u> u_0 <u>is a function such that</u> $(u_0^{m-1})_{xx} \ge 0$ <u>in</u> $\mathcal{D}'(\mathbb{R})$ <u>then</u> $(u(x,t)^{m-1})_{xx} \ge 0$ <u>in</u> $\mathcal{D}'(\mathbb{R})$ <u>for every</u> $t > 0$.

PROPERTY S3. <u>Given two solutions</u> $u, \hat{u}$ <u>with initial data</u> $u_0, \hat{u}_0$ <u>we have for every</u> $t > 0$ <u>for which both are defined</u>

(1.2) $$\int (u(x,t) - \hat{u}(x,t))_+ dx \le \int (du_0(x) - d\hat{u}_0(x))_+ ,$$

<u>where</u> $(\cdot)_+ = \max(\cdot, 0)$.

We remark that Property S3 implies in particular the pointwise comparison result; cf. for Property S1 [3], for Property S2 [1, Lemma 2] and for Property S3 [8], [9]. For the next property we refer to our work [19].

PROPERTY S4 (<u>Shifting-Comparison Lemma</u>). <u>Let</u> $u, \hat{u}$ <u>be solutions of</u> (P) <u>under conditions</u> (H1)-(H3). <u>If</u> $u_0, \hat{u}_0$ <u>satisfy</u>

(1.4) $$\int_x^\infty du_0(x) \le \int_x^\infty d\hat{u}_0(x)$$

<u>for every</u> $x \in \mathbb{R}$ <u>then for every</u> $t > 0$ <u>where both are defined we have</u>

(1.5) $$\int_x^\infty u(x,t)dx \le \int_x^\infty \hat{u}(x,t)dx .$$

COROLLARY S1. <u>Under the above assumptions if</u> $\zeta(t), \hat{\zeta}(t)$ <u>are the interfaces defined in</u> (0.2) <u>and</u> $t > 0$ <u>is as above then</u>

(1.6) $$\zeta(t) \le \hat{\zeta}(t) .$$

NOTATION. We shall use the notation $u_0 \prec \hat{u}_0$ or $\hat{u}_0 \succ u_0$ meaning that (1.4) holds. Conclusion (1.5) is then written as $u(\cdot,t) \prec \hat{u}(\cdot,t)$.

1.3. Some explicit solutions.

The following solutions will play an important role in the sequel as the models with which we compare other solutions. First we consider the solutions $w(x,t;M)$ of (P) with $w(x,0;M) = M\delta(x)$ where $M > 0$ and δ is Dirac's delta function. They are given by

$$(1.7) \qquad w(x,t;M) = t^{-\frac{1}{m+1}}\left(C - \frac{m-1}{2(m+1)m} \cdot \frac{x^2}{t^{2/(m+1)}}\right)_+^{\frac{1}{m-1}},$$

cf. [7], where C and M are related by

$$(1.8) \qquad M = a_m C^{\frac{m+1}{2(m-1)}}, \quad \text{with} \quad a_m = \left(\frac{2m(m+1)}{m-1}\right)^{\frac{1}{2}} B\left(\frac{m}{m-1}, \frac{1}{2}\right)^{(*)}$$

the right interface of $w(x,t;M)$ is given by $x = r(t)$, where

$$(1.9) \qquad r(t) = \left(\frac{2m(m+1)}{m-1} C\right)^{1/2} t^{1/(m+1)} = c_m M^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}},$$

with $c_m = \left(\frac{2m(m+1)}{m-1}\right)^{\frac{1}{m+1}} \cdot B\left(\frac{m}{m-1}, \frac{1}{2}\right)^{-\frac{m-1}{m+1}}$.

The solutions $w(x,t;M)$ serve as a model of solutions with L^1-data, cf. [19]. For solutions that blow up in finite time we shall use as model the family $z(x,t;T,C)$ defined in Q_T, $T > 0$ by

$$(1.10) \qquad z(x,t;T,C) = (T-t)^{-\frac{1}{m-1}}\left(\frac{m-1}{2m(m+1)} \cdot \frac{x^2}{(T-t)^{2/(m-1)}} + C\right)_+.$$

C can be any real number. If $C > 0$ z is always positive. If $C < 0$ z vanishes in the

(*) $B(\cdot,\cdot)$ is Euler's beta function.

region $|x| < r(T - t)$ with r defined as in (1.9). In this case we can consider the restrictions

$$z_-(x,t;T,C) = z(x,t;T,C)H(-x) \tag{1.11}$$

$$z_+(x,t;T,C) = z(x,t;T,C)H(x) \tag{1.12}$$

where $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x < 0$. If $C < 0$ z_+, z_- are solutions of (P) and in fact the right-interface of z_- is the curve $x = -r(T - t)$, $0 < t < T$.

We shall write $z(x,t;T)$, $z_\pm(x,t;T)$ instead of $z(x,t;T,0)$, $z_\pm(x,t;T,0)$.

1.4. Properties of the interface.

Let u be the solution of (P) under conditions (H1), (H2), (H3) and let $\zeta(t)$ be its interface as in (0.2). We have

PROPERTY I1. $\zeta(t)$ is finite and nondecreasing for $0 < t < T^*$.

PROOF. It is nondecreasing since Property S1, (ii) implies that if $u(x,t) > 0$ and $\bar{t} > t$, then $u(x,\bar{t}) > 0$.

To see that it is finite we remark that by Properties (H2), (H3) there exist constants $C_1, C_2 > 0$ such that

$$\int_x^\infty du_0(x) \le C_1(|x| + C_2)^{\frac{m+1}{m-1}}, \tag{1.13}$$

hence there exists $T_1 > 0$ such that $u_0 \prec z_-(x - C_2, 0; T_1)$ and the shifting comparison lemma implies that

$$\zeta(t) \le C_2 \quad \text{for} \quad 0 < t < T_1 . \tag{1.14}$$

It is clear from Theorem A (iv) that $T^* \ge T_1$. In case $T^* > T_1$ we can repeat the argument above up to any time $T < T^*$ using the fact $u \in L^\infty_{loc}([0,T^*),X)$. □

We can now define the waiting-time t^* as in the Introduction. We have $\zeta(t) = 0$ if $0 \le t \le t^*$ and $\zeta(t) > 0$ if $t^* < t < T^*$. We shall show in Section 3 that t^* is finite. We recall that the local velocity of a solution is defined in the set $\{(x,t): u(x,t) > 0\}$ by $V(x,t) = -(\frac{m}{m-1}\, u^{m-1})_x$, cf. e.g. [1]. If $t^* < T^*$ we have

PROPERTY I2. $\zeta \in C^1[t^*,T^*)$ <u>and for</u> $0 < t < T^*$ <u>the limit</u>

$$(1.15) \qquad \lim_{\substack{x \to \zeta(t) \\ u(x,t)>0}} V(x,t) = V(\zeta(t),t)$$

<u>exists and equals</u> $\zeta'(t+)$ <u>if</u> $t > 0$. <u>Moreover</u>

$$(1.16) \qquad \zeta''(t) + \frac{m}{(m+1)t}\zeta' > 0 \quad \text{in} \quad \mathcal{D}'(t^*,T^*) ,$$

<u>therefore</u> $\zeta'(t)t^{m/(m+1)}$ <u>is nondecreasing and</u> $\zeta'(t) > 0$ <u>if</u> $t > t^*$.

PROOF. (1.15) was proved in [1] and [16] for solutions with continuous, compactly-supported initial data; $\zeta \in C^1$ is proved in [10] and for (1.16) cf. [10] and [19]. The essential of the proofs remains unchanged using the properties already quoted and the remark that Theorem A, (ii) and Property S1, (i) imply that for every $0 < t < T^*$ $V(x,t)$ is a locally bounded function of x. □

PROPERTY I3. <u>If</u> u_0 <u>is a function such that</u> $(u_0^{m-1})_{xx} > 0$ <u>in</u> $\mathcal{D}'(\mathbb{R})$ <u>then</u> $\zeta(t)$ <u>is a convex function of</u> t, $0 < t < t^*$.

PROOF. By Property S2 $V(x,t)$ is a nonincreasing function of x for every $t > 0$. This means that $V(x,\bar{t}) > V(\zeta(\bar{t}),\bar{t}) = k > 0$, therefore if $k > 0$ the "constant-velocity front"

$$(1.17) \qquad \bar{u} = \left(\frac{m-1}{m} k[k(t-\bar{t}) - (x - \zeta(\bar{t}))]_+\right)^{\frac{1}{m-1}}$$

is a solution of (P) in $\mathbb{R} \times (\bar{t},T^*)$ such that $\bar{u} < u$. Hence for every $t > \bar{t}$

$$(1.18) \qquad \zeta(t) > \zeta(\bar{t}) + \zeta'(\bar{t})(t - \bar{t}) .$$

This means that ζ is convex. We remark that when $\bar{t} = t^*$ we take $\zeta'(\bar{t})$ to mean $\zeta'(\bar{t}+)$. □

2. BLOW-UP TIME

In this section we estimate the blow-up time $T^* = T^*(u)$ of any solution u of (P) such that u_0 satisfies (H1) and (H2)-(H3) is not necessary - in terms of $L_0 = L(u_0)$ defined by

$$(2.1) \qquad L_0 = \limsup_{|x|\to\infty} |x|^{-\frac{m+1}{m-1}} M(x) ,$$

where $M(x) = |\int_0^x du_0(x)|$. Then $0 \le L_0 < \infty$. It was proved in [9] that $T^* < \infty$ if and only if $L_0 > 0$ as we said.

We begin by showing that T^* depends only on the behaviour of u_0 for very large $|x|$: for any $a \in \mathbb{R}$ we define u_a^1 as the solution of (P) with initial value $u_0^1(x;a) = u_0(x) \cdot \chi((-\infty,a])$, i.e. $u_0^1(x;a)$ coincides with u_0 on $(-\infty,a]$ and vanishes on (a,∞). Likewise we define u_b^2 for some $b \in \mathbb{R}$ as the solution of (P) with initial value $u_0^2(x;b) = u_0(x) \cdot \chi([b,\infty))$. Then we have

PROPOSITION 2.1. <u>All the solutions</u> u_a^1, $a \in \mathbb{R}$, <u>have the same blow-up time</u> T_1^*. <u>Likewise the family</u> $\{u_b^2\}$ <u>has a common blow-up time</u> T_2^*. <u>Finally</u>

$$(2.2) \qquad T^*(u) = \min(T_1^*, T_2^*) .$$

PROOF. By the maximum principle, cf. (1.2), we have for every $a' < a$, $b < b'$:

$$(2.3.a) \qquad T^*(u) \le T^*(u_a^1) \le T^*(u_{a'}^1)$$

$$(2.3.b) \qquad T^*(u) \le T^*(u_b^2) \le T^*(u_{b'}^2) .$$

The fact that $T^*(u_a^1) = T^*(u_{a'}^1)$ is a consequence of (1.2) and Theorem A, (iv): (1.2) implies that for $t < T^*(u_a^1)$ $u_a^1 \ge u_{a'}^1$ and $\int (u_a^1 - u_{a'}^2)_+ dx$ is bounded by a constant that does not depend on t. Therefore $|||u_a^1(\cdot,t)|||_1$ is bounded as long as $|||u_{a'}^1(\cdot,t)|||_1$ is. By virtue of Theorem A (ii), (iv) this implies that $T^*(u_a^1) = T^*(u_{a'}^1)$.

The same argument proves that $T^*(u_b^2) = T^*(u_{b'}^2)$.

To end the proof we have to show that if $T = \min(T_1^*, T_2^*)$, then u is defined for $0 < t < T$. For this we take an $\varepsilon > 0$ and prove that the supports of $u_a^1(\cdot,t)$ and

$u_b^2(\cdot,t)$ do not meet for $0 < t < T - \varepsilon$ if $a \ll 0$ and $b \gg 0$. Assuming that this is true we conclude as follows: $\bar{u}(x,t) = u_a^1(x,t) + u_a^2(x,t)$ is then the solution to (P) in the domain $Q_{T-\varepsilon}$ with initial data $u_0 - u_0 \cdot \chi((a,b))$. From (1.2) we deduce that $u(x,t)$ is defined in $Q_{T-\varepsilon}$ and that

$$\int u(x,t)dx \le \int \bar{u}(x,t)dx + \int_{(a,b)} du_0(x) \tag{2.4}$$

for any $0 < t < T - \varepsilon$. Now let $\varepsilon \to 0$ to get $T^*(u) \ge T$.

We control finally the supports of u_a^1 and u_b^2. Let us begin with u_a^1: by Theorem A (ii) there exists a constant $C > 0$ (that depends on ε) such that for every $a < 0$, $0 < t < T - \varepsilon$ and $x < -1$

$$\int_x^\infty u_a^1(s,t)ds \le C|x|^{\frac{m+1}{m-1}} . \tag{2.5}$$

Now we observe that if we set $v_a(x,t) = z_-(x - a/2,t;\tau)$ with

$$\tau = \left(\frac{m-1}{m+1}\right)^m (8mC)^{-1} , \tag{2.6}$$

then (2.5) implies that $u_a^1(x,0) \prec v_a(x,0)$ for every $a < -1$ so that Corollary S1 of Section 2 implies that $u_a^1(x,t) = 0$ for every $x > a/2$ and $0 < t \le \min(\tau,T - \varepsilon)$. Let now N be the least integer $\ge (T - \varepsilon)/\tau$ and set $\bar{u} = u_{a'}^1$, with $a' = -2^N$. If $T - \varepsilon \le \tau$, $N = 1$ and we have proved that $\bar{u}(x,t)$ vanishes in $(-1,\infty) \times (0,T - \varepsilon)$. If $T - \varepsilon > \tau$ we can repeat the argument at $t = \tau$ with $v = z_-(x - a^{1/4},t - \tau;\tau)$ to conclude that $\bar{u}(x,t) = 0$ in $(a^{1/4},\infty) \times (0,\min(2\tau,T - \varepsilon))$. By induction it follows that $\bar{u}(x,t) = 0$ for $x > -1$ and $0 < t < T - \varepsilon$ in any case.

In the same way we can prove that $u_{b'}^2$, with $b' = 2^{N'}$ and N' defined similarly to N vanishes in $(-\infty,1) \times (0,T - \varepsilon)$. This completes the proof. □

The preceding result allows to reduce the study of the blow-up time to solutions satisfying (H1), (H2), (H3). In this case

$$L_0 = \limsup_{x\to-\infty} |x|^{-\frac{m+1}{m-1}} \int_x^\infty du_0(x) \ . \tag{2.7}$$

The main result of this section is

THEOREM 1. T^* *is infinite if and only if* $L_0 = 0$. *If* $L_0 > 0$ *we have*

$$\frac{T_m}{L_0^{m-1}} < T^* < \frac{\theta_m}{L_0^{m-1}} \tag{2.8}$$

where $T_m = \left(\frac{m-1}{m+1}\right)^m \cdot \frac{1}{2m}$ *and* $\theta_m = c_m^{-(m+1)} = \frac{m-1}{2m(m+1)} \cdot B\left(\frac{m}{m-1}, \frac{1}{2}\right)^{m-1}$ (cf. (1.9)).
In case L_0 *is actually the* limit *as* $x \to -\infty$ *or as* $x \to \infty$ *then*

$$T^* = T_m/L_0^{m-1} \ . \tag{2.9}$$

PROOF. By virtue of Proposition 2.1 we can assume that $u_0 \equiv 0$ on $(0,\infty)$.

Let $\varepsilon > 0$. There exists a constant $C = C_\varepsilon > 0$ such that $M(x) < (L_0 + \varepsilon)|x|^{(m+1)/(m-1)}$ if $x < -C$. Therefore there exists a constant $K > 0$ such that $u_0(x) \prec z_-(x - K, 0; T_\varepsilon)$ where $T_\varepsilon = T_m(L_0 + \varepsilon)^{1-m}$. It follows then from Property S4, §2, that $T^* > T_\varepsilon$ (and that for every $0 < t < T_\varepsilon$, $u(\cdot,t) \prec z_-(\cdot - K, t; T_\varepsilon)$). Letting $\varepsilon \to 0$ we obtain the left-hand inequality of (2.8).

In case L_0 is not only a lim sup but the limit as $x \to -\infty$ of the expression in (2.7) we can repeat the argument now to find a $K < 0$ such that $u_0(x) \succ z_-(x - K, 0; T_{-\varepsilon})$ where obviously $T_{-\varepsilon} = T_m(L_0 - \varepsilon)^{1-m}$. It follows that $T^* < T_\varepsilon$ hence as $\varepsilon \to 0$, $T^* < T_m L_0^{1-m}$.

We prove next the second inequality of (2.8): we choose a point $\bar{x} < -1$, move the mass in $[\bar{x},0]$ at time $t = 0$ to the point $\bar{x}$ and consider the solution $\bar{u}(x,t) = w(x - \bar{x}, t; M(\bar{x}))$ with initial data $\bar{u}_0(x) = M(\bar{x})\delta(x - \bar{x})$: we have for every $x \in \mathbb{R}$, $t > 0$

$$\bar{u}(x,t) = \left(\frac{m-1}{2m(m+1)t}\right)^{\frac{1}{m-1}} \left[c_m^2 M(\bar{x})^{\frac{2(m-1)}{m+1}} t^{\frac{2}{m+1}} - x^2\right]_+^{\frac{1}{m-1}} . \tag{2.10}$$

There is a sequence $\bar{x}_n \to -\infty$ such that, given $\varepsilon > 0$, $M(\bar{x}_n) > (L_0 - \varepsilon)|\bar{x}_n|^{(m+1)/(m-1)}$ if n is large enough, $n > n_\varepsilon$. Let us set

$$\bar{t}_\varepsilon = \frac{(1+\varepsilon)^{\frac{m+1}{2}}}{c_m^{m+1}(L_0 - \varepsilon)^{m-1}}, \tag{2.11}$$

for all large $n > n_\varepsilon$ we have:

$$c_m M(\bar{x}_n)^{\frac{m-1}{m+1}} t_\varepsilon^{\frac{1}{m+1}} > (1+\varepsilon)^{1/2}|\bar{x}_n| . \tag{2.12}$$

Hence for some $C = C_m > 0$ we have with $\bar{x} = \bar{x}_n$, $n > n_\varepsilon$

$$\bar{u}^{m-1}(x,t) > C|\bar{x}_n|^2 \varepsilon t_\varepsilon^{-1} \tag{2.13}$$

if $-1 < x < 0$. Since by construction we have $M(\bar{x}_n)\delta(x - x_n) \prec u_0(x)$ we conclude that for every $t < T^*$, $\bar{u}(\cdot,t) \prec u(\cdot,t)$ and in particular

$$\int_{-1}^{\infty} u(x,t)dx > \int_{-1}^{0} \bar{u}(x,t)dx . \tag{2.14}$$

In case $T^* > t_\varepsilon$ for an $\varepsilon > 0$ we can use (2.13) to estimate the right-hand side of (2.14) and let $n \to \infty$ to conclude that the integral $\int_{-1}^{\infty} u(x,t_\varepsilon)dx = \infty$, a contradiction. Hence $T^* < t_\varepsilon$ and letting $\varepsilon \to 0$ the result follows.

REMARK. The accuracy of estimate (2.8) depends on the ratio

$$\mu_m = \frac{\theta_m}{T_m} = \left(\frac{m+1}{m-1}\right)^{m-1} \cdot B\left(\frac{m}{m-1}, \frac{1}{2}\right)^{m-1} . \tag{2.15}$$

μ_m approaches 1 as $m \downarrow 1$. Indeed $\mu_{1+\varepsilon} = 1 + O(\varepsilon \lg \varepsilon)$ as $\varepsilon \downarrow 0$. On the contrary μ_m grows like 2^m as $m \to \infty$. $\mu_2 = 4$.

3. ON THE WAITING TIME

In the sequel $u(x,t)$ is the solution of (P) under assumptions (H1), (H2), (H3). Without loss of generality we set $\zeta(0) = 0$. We discuss in this section the existence of a positive waiting time and give estimates for it in terms of u_0.

In [2] Aronson constructed an example of solution with smooth initial data having a positive waiting time that he explicitly computed. To be specific if $u_0^{m-1}(x) = \cos^2 x$ for $-\pi/2 < x < \pi/2$, $u_0^{m-1}(x) = 0$ otherwise, he proved that $t^* = (m-1)/2m(m+1)$ and at that time the second derivative $(u_0^{m-1}(x,t))_{xx}$ blows up at $x = \pm\pi/2$. Knerr discussed in [16] (under the simplifying assumptions that u_0 is continuous, positive in a bounded interval (a,b), $a < b$, and zero outside) the waiting time t^* in terms of the behaviour of $p_0(x) = \frac{m}{m-1} u^{m-1}$ near the endpoint b: thus if $p_0(x) < C(b-x)^2$ for some $C > 0$ and all x near b then $t^* > 0$; on the contrary if $p_0(x) > C(b-x)^\alpha$ with C,x as before and $\alpha < 2$ then $t^* = 0$.

In [5] Aronson, Caffarelli an Kamin prove the following result (adapted to our notation):

THEOREM C: <u>Let</u> u <u>be a solution of</u> (P), <u>let</u> $p = \frac{m}{m-1} u^{m-1}$ <u>and assume that</u> $u_0 \in L^1_{loc}(\mathbb{R})$ <u>and that</u> $p_0(x) \equiv p(x,0) = 0$ <u>for</u> $x > 0$. <u>If</u> $p_0(x) = \alpha x^2 + o(x^2)$ as $x \uparrow 0$ <u>and</u> $p_0(x) \le \beta x^2$ <u>in</u> $\mathbb{R}^-$ <u>for some constants</u> $\alpha, \beta > 0$ <u>then</u>

$$\frac{1}{2(m+1)\beta} \le t^* \le \frac{1}{2(m+1)\alpha} . \tag{3.1}$$

COROLLARY A: <u>Under the above hypotheses if</u> $\alpha = \beta$ <u>then</u>

$$t^* = \frac{1}{2(m+1)\alpha} . \tag{3.2}$$

In this section we give a necessary and sufficient condition for the existence of a positive waiting time as well as an estimate of t^* in terms of $M(x)$. Notice that under hypothesis (H3) $M(x) = 0$ if $x > 0$ and

$$M(x) = \int_x^0 du_0(x) \quad \text{if} \quad x < 0 . \tag{3.3}$$

THEOREM 2. I) t^* <u>is positive if and only if</u>

$$\text{(3.4)} \qquad \limsup_{x\to 0} M(x)|x|^{-\frac{m+1}{m-1}} < \infty .$$

II) <u>More precisely if</u> $B = \sup_{x<0} M(x)|x|^{-\frac{m+1}{m-1}} < \infty$ <u>then</u>

$$\text{(3.5)} \qquad \frac{T_m}{B^{m-1}} < t^* < \frac{\theta_m}{B^{m-1}}$$

<u>where</u> T_m, θ_m <u>are the same constants as in Theorem 1.</u>

III) <u>If</u> $A = \liminf_{x\to 0} M(x)|x|^{-\frac{m+1}{m-1}}$ <u>is positive then</u>

$$\text{(3.6)} \qquad t^* < \frac{T_m}{A^{m-1}} .$$

COROLLARY 3.1. <u>If</u> u_0 <u>is such that the supremum of</u> $M(x)|x|^{-\frac{1+m}{m}}$ <u>is obtained as the limit when</u> $x \to 0$ <u>then</u>

$$\text{(3.7)} \qquad t^* = \frac{T_m}{B^{m-1}} .$$

REMARKS. 1) Since for $p_0(x) = bx^2$ we have

$$\text{(3.8)} \qquad M(x) = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} \cdot \frac{m-1}{m+1} \cdot b^{\frac{1}{m-1}} |x|^{\frac{m+1}{m-1}} ,$$

under the hypothesis $p_0(x) < \beta x^2$ the left-hand inequality of (3.5) gives precisely $(2(m+1)\beta)^{-1} < t^*$. Also Corollary A is implied by Corollary 3.1, the conditions being in our case far less restrictive.

2) For the accuracy of (3.5) see Remark at the end of Section 2.

PROOF OF THEOREM 2. II) Assume that $B < \infty$. We compare $u(x,t)$ with $\bar{u}(x,t) = z(x,t;\tau)$. It is immediate that $u_0 \prec \bar{u}_0$ if $\tau < T_m B^{1-m}$. Therefore we conclude from Theorem 1 that $T^* > \tau$ and from the Shifting Comparison that for $0 < t < \tau$, $\zeta(t) < \bar{\zeta}(t) = 0$, hence $t^* > \tau$. This proves (3.5), left.

For the upper bound in (2.5) we compare $u(x,t)$ with the solution $\tilde{u}(x,t) = w(x - \bar{x}, t; M(\bar{x}))$ for an $\bar{x} < 0$. Since it is clear that $\tilde{u}_0 \prec u_0$ we have for every $0 < t < T^*$, $\tilde{\zeta}(t) \le \zeta(t)$. But since

$$\tilde{\zeta}(t) = \bar{x} + c_m M(\bar{x})^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}} \tag{3.9}$$

we conclude that $\zeta(t) > 0$ if $t > c_m^{-(m+1)} |\bar{x}|^{m+1} M(\bar{x})^{-(m-1)}$. This being true for every $\bar{x} < 0$ we can take the infimum of the expression in the right-hand side and obtain thus the desired inequality.

I) Since, because of assumption (H2) B is finite if and only if $\limsup_{x\to 0} M(x)|x|^{-\frac{m+1}{m-1}}$ is finite, (3.4) follows from (3.5).

III) We first recall that any solution $\tilde{u}(x,t)$ with initial pressure $\tilde{p}_0(x) = \alpha x^2 + o(x^2)$ and such that $p_0(x) \le \alpha x^2$ has a waiting time given by (3.2) (Corollary A).

Now for every $\varepsilon > 0$ the solution $\tilde{u}$ such that $\tilde{p}_0(x) = \alpha x^2$ if $x_\varepsilon < x < 0$ and $\tilde{p}_0(x) = 0$ otherwise, satisfies $\tilde{u}_0 \prec u_0$ if

$$(A - \varepsilon) > \left(\frac{m-1}{m+1}\right)^{\frac{1}{m-1}} \cdot \frac{m-1}{m+1} \cdot \alpha^{\frac{1}{m-1}} \tag{3.10}$$

and x_ε is small enough. Therefore $t^* \le \tilde{t}^* = (2(m+1)\alpha)^{-1} = \theta_m (A - \varepsilon)^{1-m}$. Letting $\varepsilon \to 0$ we get (3.6). □

We end the section by applying our results to a family of solutions already discussed in [5]:

EXAMPLE. We let $m = 2$ and consider the solutions $u(x,t;\theta)$ with initial data

$$u_0(x) = \begin{cases} \frac{1}{2} [(1-\theta)\mathrm{sen}^2 x + \theta\,\mathrm{sen}^4 x] & \text{if } x \in [-\pi, 0] \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

with $0 \le \theta \le 1$. Notice that since $m = 2$ $p_0(x) = 2u_0(x)$.

1) We estimate the waiting time when $\theta = 1$. In this case the results of [5] imply that $0.3174 < t^*$. We obtain more accurate estimates using Theorem 2: since the maximum of $M(x)|x|^{-3}$ for $-\infty < x < 0$ is attained at the point $x = -1.449951$ with a value $B = 0.0769886$ it follows from (3.5) that

$$0.3608 < t^* < 1.4432 \ . \tag{3.12}$$

2) Now we study the range of θ's for which formula (3.2), i.e. in this case

$$t^* = \frac{1}{6(1 - \theta)} , \tag{3.13}$$

is valid. Since for $x \approx 0$ we have $M(x)x^{-3} = (1/6)(1 - \theta) + (1/30)(4\theta - 1)x^2 + O(x^4)$, if we let $\bar{\theta} = \sup\{\theta \in [0,1]: \ (3.13) \text{ holds}\}$ we have the lower estimate $\bar{\theta} > 0.25$ as in [5]. But the upper estimate in (3.5) allows us to conclude that for θ near 1 (3.13) does not hold. Indeed this happens for every $\theta > 0.88\ldots$. Therefore

$$0.25 < \bar{\theta} < 0.88\ldots \ . \tag{3.14}$$

4. MORE SELF-SIMILAR SOLUTIONS

To give exact rates of growth of the interface as $t \uparrow \infty$ or $t \downarrow 0$ we need a suitable family of models that we construct in this section.

For every α, $-1 < \alpha < 2/(m-1)$, we let $w_\alpha(x,t)$ be the solution of (P) with initial condition

$$w_\alpha(x,0) = \begin{cases} (\alpha+1)|x|^\alpha & \text{if } x < 0 \\ 0 & \text{if } x > 0 . \end{cases} \tag{4.1}$$

Since the map $u \mapsto Tu$ defined by

$$Tu(x,t) = ku(Lx,Tt) \tag{4.2}$$

where k,L,T are given positive constants, transforms solutions of $u_t = (u^m)_{xx}$ into new solutions if $k^{m-1}L^2 = T$ and since $Tw_\alpha(x,0) = w_\alpha(x,0)$ if $kL^\alpha = 1$ we conclude from the uniqueness of the solutions of (P) that for every $L > 0$ we have

$$w_\alpha(x,t) = L^{-\alpha} w_\alpha(Lx; L^{2-\alpha(m-1)}t) \tag{4.3}$$

for every $x \in \mathbb{R}$, $t > 0$. In particular if we fix $t > 0$ and then choose L such that $L^{2-\alpha(m-1)}t = 1$ we deduce that w_α can be represented in the form

$$w_\alpha(x,t) = t^{\alpha\gamma} f_\alpha(xt^{-\gamma}) \quad \text{in } Q = \mathbb{R} \times (0,\infty) , \tag{4.4}$$

with $\gamma = (2 - \alpha(m-1))^{-1} > 0$. Therefore w_α is a self-similar solution.

It is easy to see that $f_\alpha(\xi) = w_\alpha(\xi,1)$ is a nonnegative solution of the second-order differential equation

$$(f^m)''(\xi) = \alpha\gamma f(\xi) - \gamma\xi f'(\xi) \qquad ({}' = \tfrac{d}{d\xi}) \tag{4.5}$$

on the whole line $\xi \in \mathbb{R}$, such that $f(\xi) \to o(\xi^\alpha)$ as $\xi \to \infty$ and $f(\xi)|\xi|^{-\alpha} \to (\alpha+1)$ as $\xi \to -\infty$. The fact that there exists a unique solution of (4.5) with such behaviour as $|\xi| \to \infty$ follows from the existence and uniqueness of solutions of (P).

By Property I.1 the free boundary ζ_α of w_α is finite. If we let

$$\eta_\alpha = \zeta_\alpha(1) , \tag{4.6}$$

then $0 < \eta_\alpha < \infty$ and $\zeta_\alpha(t) = \eta_\alpha t^\gamma$. η_α depends only on α and m; in fact remarking that $M_\alpha(x) = |x|^{\alpha+1} \leq (|x|+1)^{\frac{m+1}{m-1}}$ gives by means of the comparison argument of

Property I.1

$$\eta_\alpha < T_m^{-\gamma} . \tag{4.7}$$

REMARKS. 1) Barenblatt [7] considered solutions of the form $t^\delta f(xt^{-\gamma})$ to solve the problem

$$\begin{cases} u_t = (u^m)_{xx} & \text{if } x > 0,\ t > 0 , \\ u(x,0) = 0 & \text{if } x > 0 , \\ u(0,t) = \sigma t^\delta & \text{if } t > 0 . \end{cases} \tag{4.8}$$

This leads to the study of equation (4.5) with $\gamma = \frac{1}{2}[1 + \delta(m - 1)]$ and side conditions $f(0) = \sigma$ and $f(\infty) = 0$. He considered the case $m > 1$, $\delta > 0$, $\sigma > 0$.

A detailed study of the problem

$$\begin{cases} (f^m)''(\xi) + \gamma\xi f'(\xi) = \delta f(\xi) & \text{for } \xi > 0 \\ f(0) = U > 0,\ f(\xi) \text{ bounded as } \xi \to \infty \end{cases} \tag{4.9}$$

with $m > 1$ and independent parameters $\gamma, \delta \in \mathbb{R}$ is made by Gilding and Peletier in [12], [13] (where references to related works can be found). In case $U > 0$ they prove that there exists a solution of (4.9) with compact support if $\gamma > 0$ and $2\gamma + \delta > 0$ and this solution is unique. In the particular case of (4.5) where α, γ, δ are related as above the conditions mean $\alpha > -2$. In this way we recover the solutions $w_\alpha(x,t)$ restricted to the quadrant $\{x > 0, t > 0\}$. Bounded positive solutions of (4.9) can be obtained under our conditions for $-2 < \alpha < 0$. Since the equation (4.5) is invariant under the transformation $\eta \mapsto -\eta$ we can recover so the left part, $(x < 0, t > 0)$, of $w_\alpha(x,t)$ if $\alpha < 0$.

But once we have the general existence and uniqueness theory for (P) our approach gives a very *simple proof* of the existence and properties of $w_\alpha(x,t)$ that relies on the use of the scaling-invariance of the equation.

2) When $\alpha = (m - 1)^{-1}$, $\gamma = 1$, we obtain the explicit solution

$$w(x,t) = \left[\frac{m-1}{m} c(ct - x)_+\right]^{\frac{1}{m-1}} \tag{4.10}$$

with a suitable $c > 0$. This is called a <u>constant-velocity front</u> since $\zeta(t) = ct$ and $-(\frac{m}{m-1} v^{m-1})_x = c$ whenever $v > 0$.

3) For $\alpha > (m-1)^{-1}$ the initial pressure is a convex function: $(w_\alpha(x,0)^{m-1})_{xx} > 0$ a.e. Therefore the same holds for every $t > 0$, i.e. f^{m-1} is convex and the free boundary $\eta_\alpha(t)$ is convex.

4) The limit case $\alpha = -1$ is represented by the solutions with finite mass, i.e. we define

(4.11) $$w_{-1}(x,t) = w(x,t;1) \quad \text{as in definition (1.7)}$$

and then

(4.12) $$\eta_{-1} = c_m, \quad \text{as in (1.9).}$$

Using again the transformation T we see that for $c > 0$ the functions

(4.13) $$w_{\alpha,c}(x,t) = c^{2\gamma} t^{\alpha\gamma} f(c^{-(m-1)\gamma} x t^{-\gamma})$$

are solutions of (P) with initial data $w_{\alpha,c}(x,0) = c w_\alpha(x,0)$. Their interface is given by $x = \eta_{\alpha,c}(t)$ where

(4.14) $$\eta_{\alpha,c}(t) = \eta_\alpha(c^{m-1}t)^\gamma .$$

Clearly (4.14) holds also for $\alpha = -1$.

5. BEHAVIOUR FOR SMALL t

We begin this section by showing that the behaviour of $\zeta(t)$ as $t \to 0$ depends only on the behaviour of $M(x)$ as $x \to 0$.

LEMMA 5.1. *Let* $u_1(x,t)$, $u_2(x,t)$ *be two solutions of* (P) *with initial data* $u_1^0(x), u_2^0(x)$, *mass functions* $M_1(x), M_2(x)$ *and interfaces* $\zeta_1(t), \zeta_2(t)$ *respectively. If*

$$\lim_{x\to 0} \frac{M_2(x)}{M_1(x)} = c, \quad 0 < c < \infty \tag{5.1}$$

then for every $\varepsilon > 0$ *there exists* $\tau > 0$ *such that if* $0 < t < \tau$

$$\zeta_1((c+\varepsilon)^{m-1}t) > \zeta_2(t) > \zeta_1((c-\varepsilon)^{m-1}t) . \tag{5.2}$$

PROOF. For every $\delta > 0$ there exists $x_\delta > 0$ such that $M_2(x) < (c+\delta)M_1(x)$ if $x_\delta < x < 0$. Now we use the transformation T, cf. Section 4, on the solution $\tilde{u}_2$ with initial data

$$\tilde{u}_2^0(x) = \begin{cases} u_2^0(x) & \text{if } x_\delta < x < 0 \\ 0 & \text{otherwise} . \end{cases} \tag{5.3}$$

We put $k = L = (1+\delta)$ so that $T = (1+\delta)^{m+1}$ and define $\tilde{u}^* = T\tilde{u}_2$, $\tilde{u}_0^* = T\tilde{u}_2^0$. The support of $\tilde{u}_0^*$ is contained in the interval $[x_\delta^*, 0]$, where $x_\delta^* = x_\delta(1+\delta)^{-1} > x_\delta$. Also $\tilde{M}^*(x) > \tilde{M}_2(x)$ for every x, i.e. $\tilde{u}_0^* \succ \tilde{u}_2^0$.

We now consider the solution $U(x,t)$ with initial condition

$$U_0(x) = \begin{cases} u_0(x) & \text{if } x < x_\delta \\ \tilde{u}_0^*(x) & \text{if } x_\delta^* < x < 0 \\ 0 & \text{otherwise} . \end{cases} \tag{5.4}$$

It is clear that $U_0 \succ u_2^0$, hence their interfaces $Z(t), \zeta_2(t)$, satisfy $Z(t) > \zeta_2(t)$ in their common interval of definition. But since $U_0(x) = 0$ in the interval $[x_\delta, x_\delta^*]$ for a certain time $\tau > 0$, $U(x,t)$ coincides with $\tilde{u}^*(x,t)$ if $x > x_\delta$ and $0 < t < \tau$, hence $Z(t) = \tilde{\zeta}^*(t)$.

To prove the first inequality of (5.2) we have yet to compare $\tilde{\zeta}_2^*$ and ζ_1. For this we use again T, now with $k = (c + \delta)(1 + \delta)$, $L = 1 + \delta$ and $T = (c + \delta)^{m-1}(1 + \delta)^{m+1}$. We obtain a solution $\tilde{u}_1 = Tu_1$ such that

$$\tilde{M}_1(x) = (c + \delta)M_1((1 + \delta)x) > M_2((1 + \delta)x) > \tilde{M}^*(x) \ , \tag{5.5}$$

i.e. $\tilde{u}_1^0 \succ \tilde{u}^*$, therefore $\tilde{\zeta}_1(t) > \tilde{\zeta}^*(t) = Z(t) > \zeta_2(t)$ if $0 < t < \tau$. Choosing $\delta > 0$ such that $(1 + \delta)^{m+1}(c + \delta)^{m-1} < (c + \varepsilon)^{m-1}$ this implies the desired inequality since $\tilde{\zeta}_1(t) = L^{-1}\zeta_1(Tt)$.

The second inequality can be obtained by reversing the roles of u_1 and u_2. □

The solutions $w_{\alpha,c}$ constructed in Section 4 are used to give precise growth rates for $\zeta(t)$ when t is small:

THEOREM 3. *Let for some* β, $0 < \beta < (m + 1)/(m - 1)$

$$\limsup_{x \nearrow 0} M(x)|x|^{-\beta} = c \tag{5.6}$$

with $0 < c < \infty$. *Then as* $t \to 0$

$$\lambda c^{(m-1)\gamma} < \limsup \zeta(t)t^{-\gamma} < \eta_\alpha c^{(m-1)\gamma} \tag{5.7}$$

where $\alpha = \beta - 1$, $\gamma = (2 - \alpha(m - 1))^{-1}$; η_α *(defined in (4.6)) and* $\gamma > 0$ *depend only* β *and* m.

If c *is the limit of* $M(x)|x|^{-\beta}$ *as* $x \to 0$ *then*

$$\lim_{t \to 0} \zeta(t)t^{-\gamma} = \eta_\alpha c^{(m-1)\gamma} \ . \tag{5.8}$$

PROOF. The right-hand inequality of (5.7) and (5.8) follow from Lemma (5.1) and formula (4.14) for the interfaces of $w_{\alpha,c}$.

To prove (5.7)-left we observe that there exists a sequence $x_n \to 0$ such that $M(x_n)|x_n|^{-\beta} \to c$ as $n \to \infty$. We may assume that $c > 0$, if not there is nothing to prove. We consider the solutions $\bar{u}_n(x,t) = w(x - x_n,t;M(x_n))$. It is clear that for every n, $u_0 \succ \bar{u}_n(x,0)$, therefore we have

$$\zeta(t) > \eta_{-1}M(x_n)^{\frac{m-1}{m+1}}t^{\frac{1}{m+1}} - |x_n| \ . \tag{5.9}$$

Now if we take a small ε, $0 < \varepsilon < c$, we have $M(x_n) > (c - \varepsilon)|x_n|^\beta$ for all large

$n \geq n_\varepsilon$. We remark now that the function

$$g(y) = Ay^{\mu} - y, \quad 0 < y < \infty; \quad 0 < \mu < 1, \tag{5.10}$$

takes on a maximum value at $y_\mu = (A\mu)^{1/(1-\mu)}$:

$$g(y_\mu) = \frac{1-\mu}{\mu} y_\mu . \tag{5.11}$$

Applying this result to (5.9) with $A = \eta_{-1}(c-\varepsilon)^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}}$ and $\mu = \beta(m-1)/(m+1)$ and setting $y_\mu = x_n$ we find that there exists a sequence $\{t_{n,\varepsilon}\}_n$ such that $t_{n,\varepsilon} \to 0$ as $n \to \infty$ and

$$\zeta(t_{n,\varepsilon}) \geq \lambda((c-\varepsilon)^{m-1} t_{n,\varepsilon})^{\gamma} \tag{5.12}$$

where $\lambda = \left(\frac{\beta(m-1)}{m+1}\eta_{-1}\right)^{\gamma(m+1)} \cdot (\gamma\beta(m+1))^{-1}$. Letting $\varepsilon \to 0$ we obtain (5.7)-left. □

REMARKS. 1) The first results on $\zeta(t)$ for small t seem to be those of [16] where it is proved that $\zeta(t) = O(t^{1/2})$ if $u_0 \in L^\infty(\mathbb{R})$.

The case $u_0 \in L^1(\mathbb{R})$ is studied in [19]: it is proved that $\zeta(t) < \eta_{-1}(M^{m-1}t)^{1/m+1}$ where $M = \|u_0\|_1$ and also that $\zeta(t)t^{-\frac{1}{m+1}} \to 0$ as $t \to 0$. The assumptions $u_0 \in L^p(\mathbb{R})$, $1 < p < \infty$ are also discussed.

2) If we let $\beta > 2$ in (5.6) and $c < \infty$ then $t_* > 0$ cf. Section 3. If $\beta < 2$ and $c > 0$ then $t_* = 0$.

3) If $\beta = 0$ the limit of $M(x)$ as $x \downarrow 0$ always exists and (5.8) applies.

6. BEHAVIOUR AS $t \to \infty$

In this section we assume that u is a global solution, i.e. $T^* = \infty$, and study the behaviour of $\zeta(t)$ for large t. The results parallel those of Section 5 but now the values of $M(x)$ as $x \to \infty$ are the only ones that matter:

LEMMA 6.1. *Let* u_1, u_2 *be two global solutions of* (P) *with initial data* u_1^0, u_2^0, *mass functions* M_1, M_2, *and interfaces* ζ_1, ζ_2 *respectively. If*

$$\lim_{x \to -\infty} \frac{M_2(x)}{M_1(x)} = c, \qquad 0 < c < \infty \tag{6.1}$$

then for every $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$\zeta_1((c+\varepsilon)^{m-1}t) + C_\varepsilon > \zeta_2(t) > \zeta_1((c-\varepsilon)^{m-1}t) - C_\varepsilon . \tag{6.2}$$

PROOF. For every $\varepsilon > 0$ there exists x_ε such that for $x < x_\varepsilon < 0$, $M_2(x) < (c+\varepsilon)M_1(x) = \tilde{M}_1(x)$, where $\tilde{M}_1$ is the mass of $\tilde{u}_1 = Tu_1$ and the constants in the transformation are $k = c + \varepsilon$, $L = 1$, $T = (c+\varepsilon)^{m-1}$. Therefore we have for every x

$$M_2(x) < \tilde{M}_1(x + x_\varepsilon) . \tag{6.3}$$

It follows that

$$\zeta_2(t) < \tilde{\zeta}_1(t) + |x_\varepsilon| = \zeta_1((c+\varepsilon)^{m-1}t) + |x_\varepsilon| . \tag{6.4}$$

Putting $C_\varepsilon = |x_\varepsilon|$ we obtain the first inequality. The second is similar.

REMARK 1. As in preceding sections Knerr [16] obtained the first results: Under simplifying assumptions on the initial data, cf. Section 3, he proved that $\zeta(t) = O(t^{1/(m+1)})$. In [19] very precise results are obtained when $u_0 \in L^1(\mathbb{R})$ (and satisfies (H1), (H3)): it is proved that

$$\text{i)} \quad \zeta(t)t^{-\frac{1}{m+1}} \to c_m M^{\frac{m-1}{m+1}}, \quad M = \|u_0\|_1 ,$$

$$\text{ii)} \quad \zeta'(t)t^{\frac{m}{m+1}} \to \frac{c_m}{(m+1)} M^{\frac{m-1}{m+1}}, \text{ and} \tag{6.5}$$

$$\text{iii)} \quad \zeta(t) - c_m M^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}} \to x_0 = M^{-1} \int x u_0(x)\,dx$$

x_0, the center of mass, can be finite or $-\infty$. Notice that (6.5) implies that for _every_ solution $\zeta(t)$ grows at least like $t^{1/(m+1)}$.

Using the solutions $w_{\alpha,c}$ in combination with Lemma 6.1 we obtain the following growth rates:

THEOREM 4. _Let for some_ β, $0 < \beta < (m+1)/(m-1)$

$$\limsup_{x\to-\infty} M(x)|x|^{-\beta} = c \tag{6.6}$$

with $0 < c < \infty$. _Then as_ $t \to \infty$

$$\lambda c^{(m-1)\gamma} < \limsup \zeta(t)t^{-\gamma} < \eta_\alpha c^{(m-1)\gamma} , \tag{6.7}$$

where α,γ,λ _and_ η_α _are as in Theorem 3_. _If moreover_ c _is the limit of_ $M(x)|x|^{-\beta}$ _as_ $x \to -\infty$ _then_

$$\lim_{t\to\infty} \zeta(t)t^{-\gamma} = \eta_\alpha c^{(m-1)\gamma} . \tag{6.8}$$

PROOF. It is completely similar to that of Theorem 3, only changing throughout $t \downarrow 0$, $x \uparrow 0$ into $t \to \infty$, $x \to -\infty$.

REMARKS. 2) If we allow $\beta = 2$ in (6.6) then if $c > 0$ there is blow-up in finite time. In case $\beta > 2$, $c > 0$, no solution of (P) exists. If $\beta = 0$ the limit of $M(x)$ as $x \to -\infty$ always exists, finite or infinite.

3) The case $u_0 \in L^p(\mathbb{R})$, $1 < p < \infty$ is treated in [19]. Notice that $u_0 \in L^p(\mathbb{R})$ implies $M(x) = O(|x|^\beta)$ with $\beta = \frac{p-1}{p}$ if $p < \infty$, $\beta = 1$ if $p = \infty$.

7. APPROACHING A BLOW-UP

In this section we assume that the blow-up time T^* is finite, i.e. that $M(x)|x|^{-\frac{m+1}{m-1}}$ does not tend to 0 as x goes to $-\infty$. We begin by describing the different possible behaviours of the interface $\zeta(t)$ as $t \uparrow T^*$. Let

(7.1) $$\ell^* = \lim_{t\uparrow T^*} \zeta(t), \quad v^* = \lim_{t\uparrow T^*} \zeta'(t) .$$

Both limits exist, either finite or infinite. The four cases that may occur are:

(I) $\ell^* = 0$, i.e. $t^* = T^*$ and $\zeta(t) = 0$ for $0 < t < T^*$. Example: $z_-(x,t;T^*)$.

(II) $0 < \ell^* < \infty$, $0 < v^* = \infty$.

(III) $0 < \ell^* < \infty$, $v^* = \infty$. Example: $z_-(x,t;T^*,C)$ with $C < 0$.

(IV) $\ell^* = \infty$, $v^* = \infty$.

Remark that because of (1.16) $\ell^* > 0$ implies $v^* > 0$ and $\ell^* = \infty$ implies $v^* = \infty$.

An example of type (II) is easily constructed as follows: let u be the solution with initial data

(7.2) $$u_0(x) = z_-(x + 1,0;T) + M\delta(x) ,$$

where $M,T > 0$. If T is small as compared with T, u equals exactly $z_-(x + 1,t;T) + w(x,t;M)$ in Ω_T, has blow-up time T and $\zeta(t) = r_M(t)$ for $0 < t < T$.

On the hand it follows from Theorems 1 and 2 that (I) happens when the limit of $M(x)|x|^{-(m+1)/(m-1)}$ exists as $x \to -\infty$ and equals B.

Examples of type (IV) will follow from Proposition 7.1 below.

We introduce now a useful concept, that of <u>blow-up set</u> $\Sigma = \Sigma(u)$:

(7.3) $$\Sigma = \{x \in \mathbb{R} : u(x,t) \to \infty \text{ as } t \to T^*\} .$$

Note that the limit of $u(x,t)$ as $t \uparrow T^*$ exists for every $x \in \mathbb{R}$ since $u_t > -u/(m + 1)t$, i.e. $u(x,t)t^{1/(m+1)}$ is nondecreasing in t, in Q_{T^*}. The following holds:

PROPOSITION 7.1. Σ <u>is an interval beginning at</u> $-\infty$.

PROOF. Let $\bar{x}$ be a point not belonging to Σ. For simplicity we take $\bar{x} = 0$. Since $\lim u(0,t) < \infty$ as $t \uparrow T^*$ there exists $C > 0$ such that $u(0,t) < C$ for $T^*/2 < C < T^*$. We want to prove that no $x > 0$ belongs to Σ. This is obvious if $\ell_* = 0$ hence in the sequel we assume that $\zeta(t) > 0$ for $t > T^* - \varepsilon > T^*/2$.

Consider for $T^* - \varepsilon < t < T^*$ the function $p(x) = mu^{m-1}(x,t)/(m - 1)$, p is a continuous nonnegative function on the interval $(0,\zeta(t))$ such that $p(0) < c$, $p(\zeta(t)) = 0$ and $p_{xx} > -K$ where $K = ((m + 1)T^*/2)^{-1}$. Now we take the parabola $\bar{p}(x) = \alpha - (k/2)(x - \beta)^2$ that passes through $(0,c)$ and $(\zeta(t),0)$, i.e. with

(7.4)
$$\alpha = \frac{k}{2}\left(\zeta + \frac{c}{k\zeta}\right)^2, \quad \beta = \frac{\zeta}{2} - \frac{c}{k\zeta} .$$

It is easy to see that $p(x) < \bar{p}(x)$ in $(0,\zeta(t))$, hence in particular

(7.5)
$$\zeta'(t) = -p_x(\zeta(t)) < -\bar{p}_x(\zeta(t)) = (k\zeta(t) + c\zeta(t)^{-1}) .$$

Integrating (7.5) from $T^* - \varepsilon$ we conclude that $\zeta(t), \zeta'(t)$ remain finite as $t \uparrow T^*$. Since the maximum of $p(x,t)$ in x in $[0,\zeta(t)]$ is less than α and α is bounded for $t \to T^*$ we conclude that for every $x > 0$, $x \notin \Sigma$. □

We set $b^* = \sup \Sigma$. It is clear that $-\infty < b^* < \ell^*$. Moreover from the above proof it follows

COROLLARY 7.1. *If b^* is finite then ℓ^* is finite. If also $\ell^* > b^*$ then v^* is finite.*

When u_0 is nicely behaved at $-\infty$ there is a simple formula for b^*. Indeed if we assume that the following limits exist:

(H2")
$$\lim_{x \to -\infty} M(x)|x|^{-\frac{m+1}{m-1}} = L_0, \quad 0 < L_0 < \infty ,$$

and

(H4)
$$\lim_{x \to -\infty} \left(\left(\frac{M(x)}{L_0}\right)^{\frac{m-1}{m+1}} + x\right) = c, \quad -\infty < c < +\infty ,$$

then we have

PROPOSITION 7.2. $b^* = c$.

PROOF. (i) $\underline{b^* \leq c}$. Assume that $c < \infty$. We shall prove that for every $c' > c$, $b^* \leq c'$. In fact there exists $C_1 > 0$ such that $M(x) \leq L_0(-x + c')^{(m-1)/(m+1)}$ for every $x < -C_1$. This means that by virtue of Property S3, §2,

$$\int_{-\infty}^{\infty} (u(x,t) - z_-(x - c',t;T^*))_+ dx \leq N' < \infty \tag{7.6}$$

for every $0 < t < T^*$ where N' does not depend on t. Now since $z_-(x - c',t;T^*) = 0$ for $x > c'$ we deduce that

$$\int_{c'}^{\infty} u(x,t)dx \leq N', \quad 0 < t < T^* . \tag{7.7}$$

This implies that $c' \geq b^*$ because of the following Lemma. Hence $b^* \leq c$.

LEMMA 7.1. <u>For every</u> $x_0 < b^*$ <u>and every</u> $\varepsilon > 0$ <u>we have</u>

$$\lim_{t \uparrow T^*} \int_{|x-x_0|<\varepsilon} u(x,t)dx = \infty . \tag{7.8}$$

PROOF. Since $(u^{m-1})_{xx} \geq -K$ (see proof of Proposition 7.1) we have for every $x, x_0 \in \mathbb{R}$, $u(x,t) \geq u(x_0,t) - (K/2)(x - x_0)^2$ either for $x > x_0$ or for $x < x_0$. Hence

$$\int_{|x-x_0|<\varepsilon} u(x,t)dx \geq (u(x_0,t) - \frac{k}{2}\varepsilon^2)\varepsilon . \tag{7.9}$$

As $u(x_0,t) \uparrow \infty$ as $t \uparrow T^*$ the result follows. □

(ii) $\underline{b^* \geq c}$. Arguing as above if $c > -\infty$, for every $c'' < c$ there exists $N'' > 0$ such that

$$\int_{-\infty}^{\infty} (z_-(x - c'',t;T^*) - u(x,t))_+ dx \leq N'' < \infty . \tag{7.10}$$

Now if $b^* < c''$ this implies arguing as in Proposition 7.1 that $u(x,t)$ is bounded above in $(c''',c'') \times (T^*/2,T^*)$ for any $c''' \in (b^*,c'')$. However Lemma 7.1 applied to z_-

implies that as $t \to T^*$

$$(7.11) \qquad \int_{b^*}^{c^*} z_-(x - c^*, t; T^*)dx \to \infty$$

contradicting (7.10). Therefore $b^* \geq c^*$, hence $b^* \geq c$. $\square$

Under the above assumptions the type (IV) corresponds precisely to $c = +\infty$. If $c < 0$ we have an interface of type (I) or (II): remark that in this case we can replace the lim in condition (H4) by lim sup (and the same proof implies that $b^* \leq c < 0 = \ell^*$). We remark finally that when $c = -\infty$ the blow-up set Σ is void: in this case the sequence

$$(7.12) \qquad s_n = \sup_{x<-1} |x|^{-\frac{m+1}{m-1}} \int u(x,t_n)dx$$

must diverge as $t_n \to T^*$ but the sup is taken at points $x_n \to -\infty$.

It would seem that the blow-up merely concerns the set Σ. However, the next result points out a global aspect:

Consider a solution u with initial data u_0 that blows up at time $T > 0$. Let $\{u_{0n}\}$ an increasing sequence of measures that converge to u_0 and let $\{u_n\}$ the corresponding solutions. Let ζ, ζ_n be their respective free boundaries. We choose u_{0n} so that u_n exists for all time $0 < t$. We have

PROPOSITION 7.3. (i) <u>For every</u> $(x,t) \in Q_T$ $\quad u_n(x,t) \uparrow u(x,t)$.

(ii) <u>For every</u> $0 < t < T$ $\quad \zeta_n(t) \uparrow \zeta(t)$.

(iii) <u>For every</u> $t > T$ $\quad \zeta_n(t) \uparrow \infty$ <u>and</u> $\zeta_n'(t) \uparrow \infty$.

(iv) <u>For every</u> $x \in \mathbb{R}$, $t > T$ $\quad u_n(x,t) \uparrow \infty$.

PROOF. (i) It is clear from the maximum principle that for every n, $u_n \leq u_{n+1} \leq u$ whenever they are defined. Theorem E and Prop. 1.6 of [9] prove that the sequences $\{u_n^m\}$, $\{(u_n^m)_x\}$ and $\{(u_n^m)_t\}$ are uniformly bounded in $L^2_{loc}(\mathbb{R} \times (0,T))$. Hence $\{u^n\}$ converges uniformly on compacts to a continuous solution $\bar{u}$ of $u_t = (u^m)_{xx}$ in Q_T. Its initial trace (that exists by [4]) $\bar{u}_0$ necessarily satisfies $u_{0n} \leq \bar{u}_0$. Moreover $\bar{u} \leq u$. It follows from Theorem B, §2, that $u = \bar{u}$.

(ii) Since obviously $\zeta_n(t) \leq \zeta_{n+1}(t) \leq \zeta(t)$ whenever they are defined we have to prove that for $t < T$ $\lim_n \zeta_n(t) \equiv \sigma(t) \geq \zeta(t)$. In fact if $\sigma(t_0) = \zeta(t_0) - \varepsilon$ for some $t_0 < T$ an $\varepsilon > 0$ this can only happen at a point where ζ already moves: $\zeta'(t_0) = k > 0$. Using the fact that $\zeta'(t_0) = -p_x(\zeta(t_0);t_0)$ where $p = (m/(m-1))u^{m-1}$ we conclude that $u(x,t_0) > 0$ for x near $\zeta(t_0)$. Since $u_n(x,t_0) = 0$ for every $x \geq \zeta_n(t_0) < \sigma(t_0)$ we arrive at a contradiction with (i).

(iii) This is the first interesting point. We know (Theorem A, (iv)) that $|||u(\cdot,t)|||_1 \to \infty$ as $t \to T$. Since $u_n \uparrow u$ it follows that for every n there exist an integer j_n and a point $(x_n,t_n) \in \Omega_T$ such that

$$\int_{x_n}^{\infty} u_{j_n}(x,t_n)dx \geq (n|x_n|)^{\frac{m+1}{m-1}}, \quad t_n \to T, \quad x_n < -1 . \tag{7.13}$$

We consider now the solution $\bar{u}_n(x,t) = w(x - x_n, t - t_n; n|x_n|^{\frac{m+1}{m-1}})$, defined for $t \geq t_n$. By Corollary S1 we have, since $u_{j_n} \succ \bar{u}_n$ at time t_n,

$$\zeta_{j_n}(t) \geq c_m(n|x_n|)(t - t_n)^{\frac{1}{m+1}} - x_n \quad \text{if} \quad t \geq t_n . \tag{7.14}$$

Now fix $t = T + \tau > T$ and let $n \to \infty$ in (7.14) to obtain $\zeta_n(t) \uparrow \infty$.

From (1.16) it follows that for every $t > 0$, $\zeta_n'(t)t \geq (m+1)(\zeta_n(t) - \zeta_n(0))$. Hence if $t > T$ and $n \to \infty$, $\zeta_n'(t) \to \infty$.

(iv) Let $p_n(x,t) = (m/(m-1))u_n^{m-1}$. Since $p_n \geq 0$, $(p_n)_{xx} \geq -((m+1)t)^{-1}$ and $(p_n)_x(\zeta_n(t),t) = -\zeta_n'(t)$ it follows from (iii) that for $t > T$ $\lim u_n(\zeta_n(t) - 1,t) = \infty$ as $n \to \infty$. The conclusion $u_n(x,t) \uparrow 0$ for every $x > 0$ follows from the fact that u_n is <u>nondecreasing</u> in x for $x > 0$, $t > 0$. A proof of this property using Caffarelli's Reflection Principle is as follows: If we compare in a domain $D = (a,\infty) \times (0,\infty)$ with $a > 0$ the functions $u_n(x,t)$ and $\bar{u}_n(x,t) = u_n(2a - x,t)$ it follows from the maximum principle that $u_n < \bar{u}_n$. Now given $0 < x_1 < x_2$ and $t > 0$ take $a = 1/2(x_1 + x_2)$ to conclude that $u_n(x_1,t) > u_n(x_2,t)$.

To prove that $u_n(x,t) \uparrow 0$ even for $x < 0$ we consider the solutions $\hat{u}_n$ with initial data $\hat{u}_{0n} = u_{0n} \cdot \chi(-\infty,a)$ with $a < 0$. They approximate the solution $\hat{u}$ with $\hat{u}_0 = u_0 \cdot \chi(-\infty,a)$. Since $\hat{T} = T$ we apply the above to conclude that $\hat{u}_n(x,t) \uparrow \infty$ for every $x > a$, $t > T$. But $\hat{u}_n < u_n$.

8. OTHER INTERFACES

If u is a solution of (P) under conditions (H1)-(H3) and $u_0(x) = 0$ for $x < a$ then an outer left-interface appears

$$\zeta_{left}(t) = \inf\{x : u(x,t) > 0\}, \qquad t > 0 . \tag{8.1}$$

The properties of ζ_{left} are completely similar to those of $\zeta(t)$. Since $u_0 \in L^1(\mathbb{R})$ the asymptotic behaviour as $t \to \infty$ is covered in [19].

Also an inner free boundary Γ_{in} may appear: it is the part of the boundary Γ of $\Omega = \{(x,t); x \in \mathbb{R}, 0 < t < T^*$ and $u(x,t) > 0\}$ in Q_{T^*} not contained in $x = \zeta(t)$ or $x = \zeta_{left}(t)$. As explained in [19] it consists of an at most countable number of locally Lipschitz arcs beginning at $t = 0$. Cf. for other details [19].

ACKNOWLEDGEMENTS

The author is grateful to D. G. Aronson for many discussions and suggestions. C. Kenig kindly provided some unpublished results.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2538	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE INTERFACES OF ONE-DIMENSIONAL FLOWS IN POROUS MEDIA		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Juan L. Vázquez		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE July 1983
		13. NUMBER OF PAGES 34
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

flows in porous media
interfaces
blow-up time
waiting time
asymptotic behaviour

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The solutions of the equation $u_t = (u^m)_{xx}$ for $x \in \mathbb{R}$, $0 < t < T$, $m > 1$, where $u(x,0)$ is a nonnegative Borel measure that vanishes for $x > 0$ (and satisfies a growth condition at $-\infty$) exhibit a finite, monotone, continuous interface $x = \zeta(t)$ that bounds to the right the region where $u > 0$. We perform a detailed study of ζ: initial behaviour, waiting time, behaviour as $t \to \infty$. For certain initial data the solutions blow up in a finite time T^*: we calculate T^* in terms of $u(x,0)$ and describe the behaviour of ζ as $t \uparrow T^*$.

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